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SUBSPACES WITHOUT THE APPROXIMATION PROPERTY

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ABSTRACT

It is proved that the Banach space l_p with $1 \leqq p < 2$ contains a subspace without AP (the case $2 < p \leqq \infty$ follows from the Enflo's construction and also from the present one). The result generalizes to the following one: if the supremum of types of X is strictly less than 2 or if the infimum of cotypes of X is strictly more than 2 then X contains a subspace without AP.

A Banach space Z has the *compact approximation property* (CAP) if for every compact $K \subset Z$ and for every $\varepsilon > 0$ there exists a compact operator $T: Z \to Z$ such that $\|Tz - z\| < \varepsilon$ for every $z \in K$.

Clearly, CAP is a (formally) weaker property than the Grothendieck's approximation property (it is, however, not known if the two properties are different).

The following criterion for a Banach space not to have CAP is a modification of Enflo's original one [1].

Throughout this paper we shall denote $I_n = \{2^n + 1, 2^n + 2, \cdots, 2^{n+1}\}$.

PROPOSITION. *A Banach space Z does not have CAP if there exist bounded sequences $z_n^* \in Z^*$, $z_n \in Z$ and finite subsets $F_n \subset Z$, $n = 1, 2, \cdots$ so that*

0° $z_n^* z_n = 1$ *for all* n,

1° $z_n^* \xrightarrow{w^*} 0$,

2° *for every* $T: Z \to Z$,

$$\left| 2^{-n} \sum_{i \in I_n} z_i^* T z_i - 2^{-n-1} \sum_{i \in I_{n+1}} z_i^* T z_i \right| \leqq \max\{\|Tf\| : f \in F_n\},$$

3° $\sum_{n=1}^{\infty} \max\{\|f\| : f \in F_n\} < \infty$.

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PROOF. We put $\beta_n(T) = 2^{-n}\Sigma_{i\in I_n} z_i^* T z_i$ and $\alpha_n = \max\{\|f\|: f \in F_n\}$. By 2° and 3°, $|\beta_n(T) - \beta_{n+1}(T)| \leqq \|T\|\alpha_n$ and therefore $\beta_n(T)$ is convergent if T is bounded. We put $\beta(T) = \lim \beta_n(T)$ for $T \in L(Z,Z)$ (= bounded linear operators). By 0°, $\beta(I) = 1$ where I is the identity on Z. Next we shall notice that β annihilates the compact operators: if T is compact, then the set $(Tz_i)_{i=1}^\infty$ is compact, therefore, since $z_n^* \xrightarrow{w^*} 0$ and z_n^* is bounded, $\lim_n \sup_i |z_n^* T z_i| = 0$, in particular $z_n^* T z_n \to 0$ and $\beta_n(T) \to 0$.

Pick now ξ_n, $n = 1, 2, \cdots$ so that $\xi_n \to \infty$ but $\Sigma \alpha_n \xi_n < \infty$. Let us take

$$K = \bigcup_{n=1}^{\infty} (\xi_n \alpha_n)^{-1} F_n \cup \{0\}.$$

K is just a sequence tending to 0 and is therefore compact.

We have, by 2° and 3°, for $T \in L(Z,Z)$,

$$\beta(T) \leqq \left(\sum \alpha_n \xi_n\right) \sup\{\|f\|: f \in K\},$$

Assume now that T is compact and that $\|Tx - x\| < \varepsilon$ for all $x \in K$. By linearity of β,

$$1 = |\beta(I - T)| \leqq \left(\sum \alpha_n \xi_n\right) \cdot \varepsilon$$

which forces ε to be bounded away from 0.

1. A subspace of l_p, $1 \leqq p < 2$, without AP

Let Δ_n be a (disjoint) partition of I_n, to be defined later on.

Let $Y = (\Sigma_n \Sigma_{A\in\Delta_n} \bigoplus l_2^A)_{l_p}$, i.e. Y is the space of all sequences $t = (t_i)_{i=1}^\infty$ such that

$$\|t\| = \left(\sum_{n=1}^{\infty} \sum_{A\in\Delta_n} \left(\sum_{i\in A} t_i^2\right)^{p/2}\right)^{1/p} < \infty,$$

equipped with the norm $\|\ \|$.

Clearly, Y is isomorphic to a subspace of l_p, $1 \leqq p \leqq \infty$ (Y is even isomorphic to l_p for $1 < p < \infty$). Let (e_i) be the unit vector basis of Y (i.e. $e_i = (\delta_{ij})_{j=1}^\infty$) and let (e_i^*) be the biorthonormal functionals of e_i; we have

$$\left\|\sum t_i e_i^*\right\|_{Y^*} = \left(\sum_{n=1}^{\infty} \sum_{A\in\Delta_n} \left(\sum t_i^2\right)^{q/2}\right)^{1/q} \qquad \left(\frac{1}{q} + \frac{1}{p} = 1\right).$$

Now define $z_i = e_{2i} - e_{2i+1} + e_{4i} + e_{4i+1} + e_{4i+2} + e_{4i+3}$ and $Z = \overline{\operatorname{span}}[z_i]_{i=1}^\infty \subset Y$. We will prove that for a suitable choice of the partitions Δ_n, Z fails to have CAP.

Notice that $(e^*_{2i} - e^*_{2i+1})_{|Z} = \frac{1}{2}(e^*_{4i} + \cdots + e^*_{4i+3})_{|Z}$ because both sides give 2 when evaluated on z_i and give 0 when evaluated on z_j, $j \neq i$. We define $z^*_i \in Z^*$ by

$$z^*_i = \tfrac{1}{2}(e^*_{2i} - e^*_{2i+1})_{|Z} \left(= \tfrac{1}{4}(e^*_{4i} + \cdots + e^*_{4i+3})_{|Z}\right).$$

We put for $T \in L(Z, Z)$,

$$\beta_n(T) = 2^{-n} \sum_{i \in I_n} z^*_i T z_i.$$

We have

$$\beta_{n+1}(T) - \beta_n(T) = 2^{-n-2} \sum_{j \in I_n} (e^*_{2j} - e^*_{2j+1}) T(e_{2j} - e_{2j+1} + e_{4j} + \cdots + e_{4j+3})$$

$$- 2^{-n-2} \sum_{i \in I_{n-1}} (e^*_{4i} + \cdots + e^*_{4i+3}) T(e_{2i} - e_{2i+1} + e_{4i} + \cdots + e_{4i+3})$$

which is equal to

$$2^{-n-2} \sum_{i \in I_{n-1}} \{ e^*_{4i} T(e_{4i} - e_{4i+1} + e_{8i} + \cdots + e_{8i+3} - e_{2i} + e_{2i+1} - e_{4i} - \cdots - e_{4i+3})$$

$$+ e^*_{4i+1} T(-e_{4i} + e_{4i+1} - e_{8i} - \cdots - e_{8i+3} - e_{2i} + e_{2i+1} - e_{4i} - \cdots - e_{4i+3})$$

$$+ e^*_{4i+2} T(e_{4i+2} - e_{4i+3} + e_{8i+4} + \cdots + e_{8i+7} - e_{2i} + e_{2i+1} - e_{4i} - \cdots - e_{4i+3})$$

$$+ e^*_{4i+3} T(-e_{4i+2} + e_{4i+3} - e_{8i+4} - \cdots - e_{8i+7} - e_{2i} + e_{2i+1} - e_{4i} - \cdots - e_{4i+3}) \}.$$

The elements in parentheses above will be called y_{4i}, y_{4i+1}, y_{4i+2}, y_{4i+3} respectively, thus

$$\beta_{n+1}(T) - \beta_n(T) = 2^{-n-2} \sum_{i \in I_{n+1}} e^*_i T y_i.$$

Now we use a trick which has been already used in [6]. Let ∇_n be a partition of I_n (∇_n will be in a sense "orthogonal" to Δ_n). We assume, for the sake of convenience, that the elements of ∇_n have equal number of elements, say m_n. Thus $\# \nabla_n = 2^n m_n^{-1}$.

We can write

$$\beta_n(T) - \beta_{n-1}(T) = 2^{-n-1} \sum_{i \in I_n} e^*_i T y_i = 2^{-n-1} \sum_{B \in \nabla_n} \sum_{i \in B} e^*_i T y_i$$

$$= 2^{-n-1} \sum_{B \in \nabla_n} 2^{-m_n} \sum_{\varepsilon} \left(\sum_{i \in B} \varepsilon_i e^*_i \right) \left(\sum_{i \in B} \varepsilon_i T y_i \right)$$

where Σ_ε means: summation with respect to all $\varepsilon_i = \pm 1$, $i \in B$.

Since $\Sigma \varepsilon_i T y_i = T \Sigma \varepsilon_i y_i$, we get

$$\begin{aligned} &| \beta_n(T) - \beta_{n-1}(T) | \\ &\leqq 2^{-n-1}(\# \nabla_n) \max \left\{ \left\| \sum_{i\in B} \varepsilon_i e_i^* \right\| \left\| T \sum_{i \in B} \varepsilon_i y_i \right\| : \varepsilon_i = \pm 1, B \in \nabla_n \right\}. \end{aligned} \tag{1}$$

In the last part of the proof we shall find partitions Δ_n, ∇_n so that for every $B \in \nabla_n$ and every choice of $\varepsilon_i = \pm 1$ and all n we have

$$\left\| \sum_{i \in B} \varepsilon_i e_i^* \right\| = m_n^{1/q}, \qquad \text{where } \frac{1}{p} + \frac{1}{q} = 1, \tag{2}$$

$$\left\| \sum_{i \in B} \varepsilon_i y_i \right\| \leqq 10 m_n^{1/2}, \tag{3}$$

$$m_n \leqq 2^{n/100}. \tag{4}$$

Let us first notice that this implies that Z fails to have CAP:

Take $F_n = \{ m_n^{1/q-1} \Sigma_{i \in B} \varepsilon_i y_i : \varepsilon_i = \pm 1, B \in \nabla_n \}$; by (1) and (2),

$$| \beta_n(T) - \beta_{n-1}(T) | \leqq \tfrac{1}{4} \max \{ \| Tf \| : f \in F_n \}$$

and, by (3) and (4),

$$\alpha_n = \max \{ \| f \| : f \in F_n \} \leqq 10 m_n^{1/q - 1 + 1/2} \leqq C 2^{Kn}$$

for some $K < 0$. Therefore the assumptions of the Proposition are satisfied.

We proceed now to construct the partitions Δ_n and ∇_n.

We denote $I_n^r = \{ i \in I_n : i \equiv r \pmod 4 \}$ for $r = 0, 1, 2, 3$. The formula $\varphi_n^r(i) = i + r$ defines a map $\varphi_n^r : I_n^0 \to I_n^r$, $r = 0, 1, 2, 3$. For $\alpha = 0, 1$ we define maps $\psi_{n,\alpha} : I_n^0 \to I_{n+1}^0$ by formulas

$$\psi_{n,\alpha}(i) = 2i + 4\alpha.$$

All the above maps are 1–1 and have disjoint ranges.

We can represent I_n^0 as $I_n^0 = E_n \times F_n$ so that

$$\# E_n \geqq 2^{n/2 - 10}, \qquad \# F_n \geqq 2^{n/2-10} \tag{5}$$

and so that the following recurrence relations are satisfied for $\alpha = 0, 1$:

$$\begin{aligned} &\forall f \in F_n \, \exists e \in E_{n+1} \quad \text{so that} \quad \psi_{n,\alpha}(E_n \times \{f\}) \subset \{e\} \times F_{n+1} \\ &\forall f \in F_n \, \exists e \in E_{n-1} \quad \text{so that} \quad E_n \times \{f\} \subset \bigcup_{\alpha=0}^{1} \psi_{n-1,\alpha}(\{e\} \times F_{n-1}) \end{aligned} \tag{6}$$

(we can do it by an easy induction: on the $n+1$st step we take either

$$E_{n+1}=\bigcup_{\alpha=0}^{1}\psi_{n,\alpha}(F_n) \text{ and } F_{n+1}=\psi_{n,0}(E_n) \text{ or } E_{n+1}=\psi_{n,0}(E_n) \text{ and}$$

$F_{n+1}=\bigcup_{\alpha=0}^{1}\psi_{n,\alpha}(E_n)$, making the choice so that (5) is still fulfilled).

Finally, we split each E_n, quite independently of what has been done until now, as

$$E_n=\prod_{r=0}^{3}E_n^r \quad \text{i.e.} \quad E_n=E_n^0\times E_n^1\times E_n^2\times E_n^3$$

demanding just that $\# E_n^r \geqq 2^{n/8-10}$ for $r=0,1,2,3$.

We set now[†]

$$\nabla_n=\left\{\varphi_n^r(E_n^r\times\{f\}) : f\in \prod_{s\neq r}E_n^s\times F_n\,;\ r=0,1,2,3\right\}$$

and

$$\Delta_n=\left\{\varphi_n^r\left(\{e\}\times\prod_{s\neq r}E_n^s\times F_n\right) : e\in E_n^r;\ r=0,1,2,3\right\}.$$

The condition (4) is obviously satisfied (for big n), to prove (2) and (3) let

$$B\in\nabla_n,\ B=\varphi_n^r(E_n^r\times\{f\}) \text{ with } f=\prod_{s\neq r}f_s\times f',\ f'\in F_n.$$

(2) is almost immediate. We claim that for every $A\in\Delta_n$, $\# A\cap B\leqq 1$. Indeed, let

$$A=\varphi_n^t\left(\{e\}\times\prod_{s\neq t}E_n^s\times F_n\right).$$

If $t\neq r$, then $A\cap B=\emptyset$. Otherwise $A\cap B=\{\varphi_n^r(e\times f)\}$. Now we have

$$\left\|\sum_{i\in B}\varepsilon_i e_i^*\right\|_{Y^*}=\left(\sum_{A\in\Delta_n}(\# A\cap B)^q\right)^{1/q}=(\#\{A\in\Delta_n : A\cap B\neq\emptyset\})^{1/q}$$

$$=(\# B)^{1/q}=m_n^{1/q}.$$

The proof of (3) is a bit more involved. A look at the formula for y_i convinces us that for every ε_i

[†] We use the convention $A\times B=B\times A$.

$$\sum_{i\in B}\varepsilon_i y_i = -\sum_{s\neq r} a_s \sum_{i\in\varphi_n^s[E_n^r\times\{f\}]}\varepsilon_i e_i + (-1)\sum_{s=0}^{3}\sum_{i\in\varphi_{n+1}^s\psi_{n,\alpha}[E_n^r\times\{f\}]}\varepsilon_i e_i \tag{7}$$
$$-\sum_{i\in\varphi_{n-1}^{\beta}\psi_{n-1,\gamma}^{-1}[E_n^r\times\{f\}]}\varepsilon_i e_i + \sum_{i\in\varphi_{n-1}^{\beta+1}\psi_{n-1,\gamma}^{-1}[E_n^r\times\{f\}]}\varepsilon_i e_i\,;$$

here a_s are respectively 1 or 2, α, β, γ are 0, 1 or 2. The point is now that, by (6), all the index sets above are contained in some elements of Δ_{n-1}, Δ_n, Δ_{n+1}, respectively. The following is a formal proof:

1° for $s\neq r$, $E_n^r\times\{f\}=f_s\times\Pi_{t\neq s,t\neq r}f_t\times f'\subset f_s\times\Pi_{t\neq s}E_n^t\times F_n$ and therefore

$$\varphi_n^s(E_n^r\times\{f\})\subset\varphi_n^s\Big(\{f_s\}\times\prod_{t\neq s}E_n^t\times F_n\Big)\in\Delta_n.$$

2° by (6), $\varphi_{n+1}^s\psi_{n,\alpha}(E_n^r\times\{f\})\subset\varphi_{n+1}^s\psi_{n,\alpha}(E_n\times\{f\})\subset\varphi_{n+1}^s(\{e\}\times F_{n+1})$ for some $e=\Pi_{s=0}^3 e_s$ with $e_s\in E_{n+1}^s$. Therefore

$$\varphi_{n+1}^s(\{e\}\times F_{n+1})\subset\varphi_{n+1}^s\Big(\{e_s\}\times\prod_{t\neq s}E_{n+1}^t\times F_n\Big)\in\Delta_{n+1}\quad\text{for } s=0,1,2,3.$$

3° by (6), $\varphi_{n-1}^{\beta}\psi_{n-1,\gamma}^{-1}(E_n^r\times\{f\})\subset\varphi_{n-1}^{\beta}\psi_{n-1,\gamma}^{-1}(E_n\times\{f\})\subset\varphi_{n-1}^{\beta}(\{e\}\times F_{n-1})$ for some $e=\Pi_{s=0}^3 e_s$ with $e_s\in E_{n-1}^s$. Therefore

$$\varphi_{n-1}^{\beta}(\{e\}\times F_{n-1})\subset\varphi_{n-1}^{\beta}\Big(\{e_\beta\}\times\prod_{s\neq\beta}E_{n-1}^s\times F_{n-1}\Big)\in\Delta_{n-1}.$$

This tedious argument shows that we can rewrite (7) as

$$\sum_{i\in B}\varepsilon_i y_i=\sum_{j=1}^{10}\pm\sum_{i\in A_j}\varepsilon_i e_i$$

where $A_1,\cdots,A_{10}$ are subsets of some elements of Δ_{n-1}, Δ_n, Δ_{n+1}, respectively and, in fact, $\# A_j=\# B$ for $j=1,\cdots,10$ (the number 10 arises since $\Sigma_{s\neq r}a_s=2+1+1=4$, $\Sigma_{s=0}^3$ gives 4 more elements and two last components give each 1). Therefore

$$\Big\|\sum_{i\in A_j}\varepsilon_i e_i\Big\|=(\# A_j)^{1/2}=(\# B)^{1/2}=m_n^{1/2}$$

and, finally,

$$\Big\|\sum {}_iy_i\Big\|\leqq 10m_n^{1/2}.$$

2. The generalisation

For an infinite dimensional Banach space X let $p(X) = \sup\{p: X \text{ has type } p\}$ and $q(X) = \inf\{q: X \text{ has cotype } q\}$. The results of Maurey and Pisier [5] combined with the work of Krivine [4] yield the following important result: *For every X, $l_{p(X)}$ and $l_{q(X)}$ are finitely representable in X.*

On the other hand, it has been observed by Figiel [2] that *if l_p has a subspace without AP and l_p is finitely representable in X then X has a subspace without AP.* (This can be also easily seen from our construction, actually we have used there only the fact that $Y = \Sigma_n \oplus (\Sigma_{A \in \Delta_n} \oplus l_2^A)_{l_p}$ where $\Sigma_n \oplus$ is *any* direct sum decompostion.)

Although we have not considered the case $p > 2$, it should be clear that the whole construction carries to this case as well, we just have to interchange the roles of Δ_n and ∇_n. As said before, the case $p > 2$ follows anyhow from Enflo's construction, as shown by Davie, Figiel and Kwapień.

In this way we obtain the result stated in the Abstract: if $p(X) < 2$ or $q(X) > 2$ then X contains a subspace without AP.

The limit case $p(X) = q(X) = 2$ has been recently settled by W. B. Johnson [3]. He proved that if $p_n \to 2$ and $K_n \to \infty$ very fast, then every subspace of $(\Sigma \oplus l_{p_n}^{K_n})_{l_2}$ has AP.

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